

## Physics 681-481; CS 483: Assignment #7

(please hand in after the lecture, Thursday, May 4)

This is the final assignment.

1. Suppose the only kinds of errors one had to worry about were bit-flip errors, but one wanted to take into account not only single bit-flips ( $\mathbf{X}_i$ ) in the code words but also double bit-flips ( $\mathbf{X}_i\mathbf{X}_j$ ,  $i \neq j$ ).

(a) Show that there is an  $n$  for which the dimension of the  $n$ -Qbit state space is just large enough to accommodate mutually orthogonal two-dimensional subspaces for the uncorrupted code words and all code words suffering either single or double bit-flip corruptions. What is that  $n$ ?

(b) Show for the  $n$  you found in (a) that there is indeed a “perfect”  $n$ -Qbit code that corrects all single and double bit-flip errors, by writing down the states that encode  $|\bar{0}\rangle$  and  $|\bar{1}\rangle$ , and writing down a set of commuting hermitian operators whose squares are unity, that preserve both codewords, and have distinct patterns of commutations or anticommutations (show this explicitly) with each of the operators associated with all the single and double bit-flip errors.

2. The figure on the last page shows a circuit that produces the 5-Qbit code words. Confirm that it does:

(a) Show first that whether  $x = 0$  or  $1$ ,  $|\bar{x}\rangle$  is invariant under the operators  $\mathbf{M}_0$ ,  $\mathbf{M}_1$ ,  $\mathbf{M}_2$ , and  $\mathbf{M}_3$  given by

$$\mathbf{M}_0 = \mathbf{Z}_1\mathbf{X}_2\mathbf{X}_3\mathbf{Z}_4, \quad \mathbf{M}_1 = \mathbf{Z}_2\mathbf{X}_3\mathbf{X}_4\mathbf{Z}_0, \quad \mathbf{M}_2 = \mathbf{Z}_3\mathbf{X}_4\mathbf{X}_0\mathbf{Z}_1, \quad \mathbf{M}_3 = \mathbf{Z}_4\mathbf{X}_0\mathbf{X}_1\mathbf{Z}_2. \quad (1)$$

Show also that  $\bar{\mathbf{Z}}|\bar{x}\rangle = (-1)^x|\bar{x}\rangle$  where  $\bar{\mathbf{Z}} = \mathbf{Z}_4\mathbf{Z}_3\mathbf{Z}_2\mathbf{Z}_1\mathbf{Z}_0$ , and that  $\bar{\mathbf{X}}|\bar{0}\rangle = |\bar{1}\rangle$  and  $\bar{\mathbf{X}}|\bar{1}\rangle = |\bar{0}\rangle$  where  $\bar{\mathbf{X}} = \mathbf{X}_4\mathbf{X}_3\mathbf{X}_2\mathbf{X}_1\mathbf{X}_0$ .

A highly efficient way to evaluate the effect of each of the above six operators on  $|\Psi\rangle$  is to enter the 1-Qbit gates of which they are composed on the extreme right of the diagram. For  $\mathbf{M}_0$ , for example, you would enter  $\mathbf{Z}$  gates on the top wire and the third wire down from the top, and  $\mathbf{X}$  gates on the wires one and two down from the top. Then bring the 1-Qbit gates all the way to the far left, *carefully* using the rules in the lower half of the figure to bring them through the cNOT gates. You also need the familiar rules  $\mathbf{HZ} = \mathbf{XH}$  and  $\mathbf{HX} = \mathbf{ZH}$  to bring them through Hadamard gates, and the rule  $\mathbf{ZX} = -\mathbf{XZ}$  to bring them through the  $\mathbf{Z}$  gates. The 1-Qbit gates that emerge on the far left from this process all turn out to act very simply on the 5-Qbit states  $|10000\rangle$  or  $|00000\rangle$ .

List the gates that this process produces on the far left for each of the six operators  $\mathbf{M}_0 \dots \mathbf{M}_3$ ,  $\bar{\mathbf{Z}}$ , and  $\bar{\mathbf{X}}$ , and confirm that their action on the states  $|10000\rangle$  and  $|00000\rangle$  do

indeed lead directly to the results stated above. [You need not report the intermediate stages of this process as you bring the operators through successive cNOT and H gates. But you will have to keep track of this yourself to arrive at the final forms of the operators after they have been moved to the left through all the gates of the circuit. It might be useful to print many copies of the figure to scribble on.]

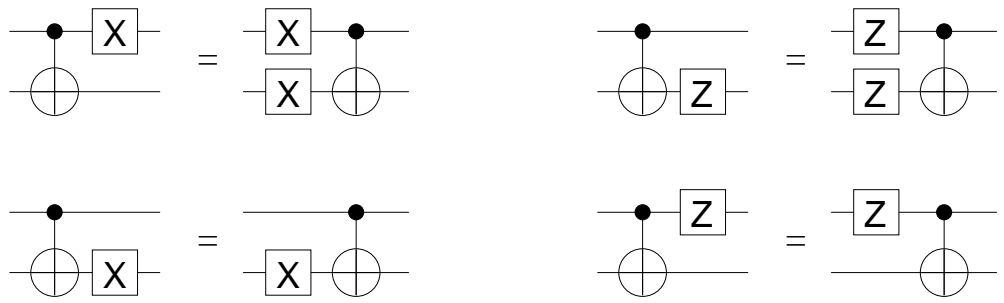
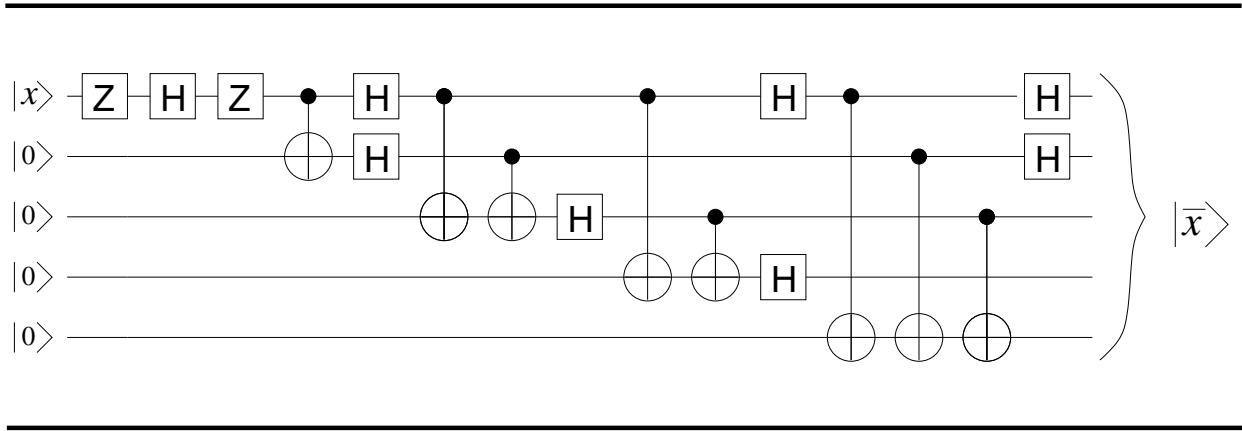
(b) Explain why the results in (a) guarantee that the two states  $|\Psi\rangle$  emerging on the left are the codeword states

$$\begin{aligned} |\bar{0}\rangle &= \frac{1}{4}(\mathbf{1} + \mathbf{M}_0)(\mathbf{1} + \mathbf{M}_1)(\mathbf{1} + \mathbf{M}_2)(\mathbf{1} + \mathbf{M}_3)|00000\rangle, \\ |\bar{1}\rangle &= \frac{1}{4}(\mathbf{1} + \mathbf{M}_0)(\mathbf{1} + \mathbf{M}_1)(\mathbf{1} + \mathbf{M}_2)(\mathbf{1} + \mathbf{M}_3)|11111\rangle. \end{aligned} \quad (2)$$

to within a common overall phase. Confirm that the phase factor is indeed 1, by evaluating the inner product of  $|00000\rangle$  with  $|\bar{0}\rangle$ . This can also be efficiently done with the help of the circuit diagram, taking advantage of the fact that many operators in the circuit can successively be replaced by the identity when evaluating  $\langle 00000|\bar{0}\rangle$ .

I found that this inner product could be quite efficiently evaluated by letting all the operators in the diagram (starting with those on the *right*) act on  $|00000\rangle$ , and taking the inner product of the resulting state with  $|00000\rangle$ ; i.e. I took advantage of the fact that if  $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$  are all hermitian (as every gate in the circuit diagram is, each being unitary and its own inverse) then

$$[\langle \phi | \mathbf{A}_1 \mathbf{A}_2 \cdots \mathbf{A}_n | \psi \rangle]^* = \langle \psi | \mathbf{A}_n \cdots \mathbf{A}_2 \mathbf{A}_1 | \phi \rangle. \quad (3)$$



The upper part of the figure shows a circuit that produces the 5-Qbit code. Acting on  $|x0000\rangle$  on the left, it produces the 5-Qbit code word  $|\bar{x}\rangle$  on the right. The lower part of the figure shows some elementary identities enabling one to move  $X$  or  $Z$  operators acting on either the control or target Qbits of a cNOT gate from the right side of the cNOT to the left side.