Notes on the transform matrix S

Question: Suppose $\hat{\mathbf{b}}$ and $\hat{\mathbf{r}}$ are two arbitrary unit vectors in $\mathbb{R}^3$. Find a unitary transform S such that

$$S\,(\hat{\mathbf{b}} \cdot \boldsymbol{\sigma})S^\dagger = \hat{\mathbf{r}} \cdot \boldsymbol{\sigma}\ .$$

Answer: S acts on $(\hat{\mathbf{b}} \cdot \boldsymbol{\sigma})$ as the adjoint representation of $SU(2)$ which is isomorphic to the 3-dimensional vector representation (rotation), via,

$$\hat{\mathbf{b}} \cdot \boldsymbol{\sigma} \mapsto \hat{\mathbf{b}}.$$

If we can find a rotation R in $\mathbb{R}^3$ which takes $\hat{\mathbf{b}}$ to $\hat{\mathbf{r}}$. Then by the isomorphism, the spin-1/2 representation of R, $S(R)$, will takes $\hat{\mathbf{b}} \cdot \boldsymbol{\sigma}$ to $\hat{\mathbf{r}} \cdot \boldsymbol{\sigma}$ by the adjoint action. So we need two steps:

- (1) Find the suitable rotation R in $\mathbb{R}^3$. There are many different choices and one easy way is the right hand screw rotation around the vector $\hat{\mathbf{b}} \times \hat{\mathbf{r}}$[1] through θ, the angle between $\hat{\mathbf{b}}$ and $\hat{\mathbf{r}}$. Notice that $0 < \theta < \pi$.
- (2) Find the spin-1/2 representation of R and hence the adjoint action. Generally, the spin-1/2 representation of right hand screw rotation around a unit vector $\hat{\mathbf{n}}$ through the angle θ is,

  $$\exp(-\frac{i\theta}{2}\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}).$$

  Expand it and use the fact $(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}) = 1$, we get

  $$\exp(-\frac{i\theta}{2}\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}) = \cos(\theta/2) - i\sin(\theta/2)\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}.$$

  For the rotation R in (1), $\hat{\mathbf{n}} = (\hat{\mathbf{b}} \times \hat{\mathbf{r}})/|\hat{\mathbf{b}} \times \hat{\mathbf{r}}|$ and the corresponding operator is

  $$S(R) = \cos(\theta/2) - i\sin(\theta/2)\frac{(\hat{\mathbf{b}} \times \hat{\mathbf{r}}) \cdot \boldsymbol{\sigma}}{|\hat{\mathbf{b}} \times \hat{\mathbf{r}}|}. \tag{1}$$

[1] Suppose $\hat{\mathbf{b}}$ and $\hat{\mathbf{r}}$ are not in the same line.

Now we replace the parameter θ. Since $0 < \theta < \pi$, both $\cos(\theta/2)$ and $\sin(\theta/2)$ are larger than 0. Therefore,

$$\cos(\theta/2) = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1+\hat{\mathbf{b}}\cdot\hat{\mathbf{r}}}{2}}$$
$$\sin(\theta/2) = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\hat{\mathbf{b}}\cdot\hat{\mathbf{r}}}{2}}.$$

And

$$|\hat{\mathbf{b}}\times\hat{\mathbf{r}}| = \sqrt{1-(\hat{\mathbf{b}}\cdot\hat{\mathbf{r}})^2}.$$

Put all above into (1), we get

$$S(R) = \frac{1}{\sqrt{2}}\left(\sqrt{1+\hat{\mathbf{b}}\cdot\hat{\mathbf{r}}} - i\frac{(\hat{\mathbf{b}}\times\hat{\mathbf{r}})\cdot\boldsymbol{\sigma}}{\sqrt{1+\hat{\mathbf{b}}\cdot\hat{\mathbf{r}}}}\right).$$

which is the transform matrix that satisfies the requirement.