Solution 9

1. **(a)** Start with the expression

$$S = \frac{1}{\sqrt{2}}\left(\sqrt{1+\hat{\mathbf{b}}\cdot\hat{\mathbf{z}}} - i\frac{\hat{\mathbf{b}}\times\hat{\mathbf{z}}\cdot\boldsymbol{\sigma}}{\sqrt{1+\hat{\mathbf{b}}\cdot\hat{\mathbf{z}}}}\right) = T_1 + T_2 \ , \tag{1}$$

where T_1, T_2 are the two terms above and $\hat{\mathbf{b}} = \mathbf{B}/B$ is a unit vector giving the local direction of the magnetic field

$$\mathbf{B} = (\alpha x - \beta z x/2)\hat{\mathbf{x}} + (-\alpha y - \beta z y/2)\hat{\mathbf{y}} + (B_0 + \beta(2z^2 - x^2 - y^2)/4)\hat{\mathbf{z}} \ , \tag{2}$$

with

$$B = B_0 + \frac{1}{2}\beta(x^2+y^2+z^2) \qquad \Longrightarrow \qquad \boldsymbol{\nabla} B = \beta\mathbf{r} = \beta(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}). \tag{3}$$

Next, using the definition

$$\mathbf{A} = S(\boldsymbol{\nabla} S^\dagger) \ , \tag{4}$$

we can say that

$$\mathbf{A} = (T_1 + T_2)(\mathbf{T}_3 + \mathbf{T}_4), \quad \mathbf{T}_3 = \boldsymbol{\nabla} T_1, \quad \mathbf{T}_4 = \boldsymbol{\nabla} T_2^\dagger. \tag{5}$$

From (2) we see that (will use notation $r=0$ instead of $x=y=z=0$ below)

$$T_2|_{r=0} = 0 \quad \text{due to} \quad \hat{\mathbf{b}}\times\hat{\mathbf{z}}\cdot\boldsymbol{\sigma} = b_y\sigma_x - b_x\sigma_y, \quad \text{and} \quad b_y|_{r=0} = b_x|_{r=0} = 0. \tag{6}$$

It is easy to see that the numerator in T_2 doesn't cancel out in combination with $\mathbf{T}_3$ or $\mathbf{T}_4$, thus we have $T_2\cdot\mathbf{T}_3|_{r=0} = T_2\cdot\mathbf{T}_4|_{r=0} = 0$.

On the other hand,

$$T_1\,\mathbf{T}_3 = \frac{1}{2}\sqrt{1+b_z}\,\boldsymbol{\nabla}\left(\sqrt{1+b_z}\right) = \frac{1}{4}\boldsymbol{\nabla} b_z = \frac{1}{4}\left(\frac{\boldsymbol{\nabla} B_z}{B} - B_z\,\frac{\boldsymbol{\nabla} B}{B^2}\right), \tag{7}$$

and using (3) where $\boldsymbol{\nabla} B|_{r=0} = 0$ and from (2) $\boldsymbol{\nabla} B_z|_{r=0} = 0$ we get $T_1\cdot\mathbf{T}_3|_{r=0} = 0$ as well. The only term left to evaluate in the definition of $\mathbf{A}$ (4) is

$$T_1\,\mathbf{T}_4 = \frac{i}{2}\sqrt{1+b_z}\,\boldsymbol{\nabla}\left(\frac{b_y\sigma_x - b_x\sigma_y}{\sqrt{1+b_z}}\right) = \frac{i}{2}\left[-\frac{(b_y\sigma_x - b_x\sigma_y)\boldsymbol{\nabla} b_z}{2(1+b_z)} + \boldsymbol{\nabla}(b_y\sigma_x - b_x\sigma_y)\right]. \tag{8}$$

Using already obtained $\boldsymbol{\nabla} b_z|_{r=0} = 0$, and similarly (from (2) and (3)) deriving $\boldsymbol{\nabla} b_x|_{r=0} = \alpha\hat{x}/B_0$ and $\boldsymbol{\nabla} b_y|_{r=0} = -\alpha\hat{y}/B_0$, we arrive at the final result

$$\mathbf{A}|_{r=0} = -i\,\frac{\alpha}{2B_0}\,(\sigma_y\hat{x} + \sigma_x\hat{y}) \ . \tag{9}$$

(b) Considering the Hamiltonian

$$H = -\frac{\hbar^2}{2M}(\mathbf{\nabla} + \mathbf{A}) \cdot (\mathbf{\nabla} + \mathbf{A}) - \mu B\, \sigma_z \ , \tag{10}$$

we have the $\mathbf{A}$ involved terms

$$2\ \mathbf{A} \cdot \mathbf{\nabla} + \mathbf{\nabla} \cdot \mathbf{A} + \mathbf{A}^2.$$

Using definitions (1) and (4) one can show that $\mathbf{\nabla} \cdot \mathbf{A} = (\alpha^2/2B_0^2 - 3\beta/4B_0)I$ (I is an identity matrix), which gives $\mathbf{\nabla} \cdot \mathbf{A} = 0$ for $\alpha = \sqrt{3B_0\beta/2}$. Anyway, $\mathbf{\nabla} \cdot \mathbf{A} + \mathbf{A}^2 = -(3\beta/4B_0)I$, is a diagonal term that does not contribute to spin flip. Thus, we can use the expression for $\mathbf{A}$ as in (9) to get

$$H_{flip} = -\frac{\hbar^2}{2M}\ 2\ \mathbf{A} \cdot \mathbf{\nabla} = i\ \frac{\hbar^2 \alpha}{2MB_0}\ (\sigma_y \frac{\partial}{\partial x} + \sigma_x \frac{\partial}{\partial y})\ . \tag{11}$$

2. Our goal is to calculate

$$\Gamma = \int d^3k \frac{V}{(2\pi)^3}\ \frac{2\pi}{\hbar}\ | < \Psi_{\mathbf{k}} | H_{flip} | \Psi_i > |^2, \tag{12}$$

where Ψ_i, $\Psi_{\mathbf{k}}$ are the initial and the final states correspondingly.

Using results from last homework, we have

$$\Psi_i \ = \ \Psi_i(\mathbf{r}) \mid \downarrow >= \left(\frac{\lambda}{\pi}\right)^{3/4} e^{-\lambda\, r^2/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad E_i = \frac{3}{2}\frac{\hbar^2 \lambda}{M} + \mu B_0, \tag{13}$$

where $\lambda = \sqrt{\mu\beta M}/\hbar$, and the final states are

$$\Psi_k \ = \ \Psi_{\mathbf{k}}(\mathbf{r}) \mid \uparrow >= \frac{e^{i\mathbf{kr}}}{\sqrt{V}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad E_k = \frac{\hbar^2 k^2}{2M} - \mu B_0. \tag{14}$$

Using initial (spin-down) and final (spin-up) spin states we get

$$<\uparrow |\sigma_x| \downarrow> \ = \ (1 \quad 0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ = \ 1, \tag{15}$$

$$<\uparrow |\sigma_y| \downarrow> \ = \ (1 \quad 0) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ = \ -i. \tag{16}$$

This leaves us with the expression

$$< \Psi_{\mathbf{k}}|H_{flip}|\Psi_i > \; = \frac{-i\hbar^2\alpha}{2MB_0}\left(\frac{\lambda}{\pi}\right)^{\frac{3}{4}} \int d^3r \, \frac{e^{i\mathbf{kr}}}{\sqrt{V}} \, \lambda \, (-ix+y) \, e^{-\lambda r^2/2}, \tag{17}$$

where putting $d^3r = dxdydz$ and $\mathbf{kr} = k_x x + k_y y + k_z z$ and we arrive at

$$< \Psi_{\mathbf{k}}|H_{flip}|\Psi_i > \; = \frac{-i\hbar^2\alpha}{MB_0}\left(\frac{\pi}{\lambda}\right)^{\frac{3}{4}} \sqrt{\frac{2}{V}} \, e^{-k^2/2\lambda} \, (k_x + ik_y), \tag{18}$$

or using spherical coordinates for $\mathbf{k}$

$$| < \Psi_{\mathbf{k}}|H_{flip}|\Psi_i > |^2 \; = \; \frac{2}{V} \, \left(\frac{\hbar^2\alpha}{MB_0}\right)^2 \left(\frac{\pi}{\lambda}\right)^{\frac{3}{2}} \, e^{-k^2/\lambda} \, k^2 \sin^2\theta, \tag{19}$$

From here we see that the angular distribution of the particles emitted from the trap does not depend on the angle φ, but on θ only.

Evaluating the integral

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin^3\theta \, d\theta = \frac{8\pi}{3}$$

we get

$$\Gamma = \frac{2\pi}{\hbar} \int \frac{V \, d\Omega}{(2\pi)^3} \, \left[k^2 \left|\frac{dk}{dE_k}\right| \, | < \Psi_{\mathbf{k}}|H_{flip}|\Psi_i > |^2\right]_{k=k(E_i)} \; = \; \frac{4}{3}\frac{\hbar\alpha^2\sqrt{\pi}}{MB_0^2}\frac{k^3 \, e^{-k^2/\lambda}}{\lambda^{3/2}} \tag{20}$$

Or introducing notations

$$\hbar\omega_s = \hbar\sqrt{\frac{\mu\beta}{M}}, \quad \hbar\omega_f = 2\mu B_0, \quad \frac{\hbar^2 k^2}{2M} = \hbar\omega_f,$$

and remembering that $\lambda = \sqrt{\mu\beta M}/\hbar$, and $\alpha = \sqrt{3B_0\beta/2}$, we get

$$\Gamma = 8\sqrt{2\pi\omega_s\omega_f} \, e^{-2\omega_f/\omega_s}, \tag{21}$$

which in its turn gives

$$\tau = \frac{1}{\Gamma} = \frac{1}{8\sqrt{2\pi\omega_s\omega_f}} \, e^{2\omega_f/\omega_s}. \tag{22}$$