Assignment 11

In this assignment we are again concerned with time-dependent effects. The first problem introduces an instance of the Golden Rule formula where the energy continuum arises from the sum over photons in the initial state. Time dependence in the second problem comes about through a tunneling process and is calculated with the WKBJ formalism developed in lecture. This assignment is due **Friday**, November 16.

1. Consider a generalization of the harmonic-in-time perturbation from the previous assignment:

$$H_{\text{int}} = e\,\mathcal{E}_i \cos(\omega_i t)\, \boldsymbol{\epsilon}_i \cdot \mathbf{r}$$

Here $\mathbf{r}$ is the electron position and $\boldsymbol{\epsilon}$ is the polarization of the electromagnetic wave. As before, this linear potential is valid in the limit where the wavelength of the wave is much larger than the size of the quantum system. The subscript i refers to the interpretation of H_{int} as being the result of a photon in the **initial** state with a particular amplitude, frequency and polarization. We are interested in the rate a quantum system makes transitions to an excited state by absorbing one of these photons. Our treatment of photons in this exercise is "semi-classical", an approach that turns out to be consistent with the true quantum dynamics.

Begin by reviewing the derivation of the Golden Rule, modified for a harmonic perturbation. The initial state $|\Psi_i\rangle$ will be the combination of an electron in the ground state 1s-orbital of a spherical cavity and a photon with the parameters specified above. There are three final states $|\Psi_f\rangle$ corresponding to the three degenerate 1p-orbitals of the electron in the same cavity. The probability of f as the final state, for an initial state i, is given by

$$|C_{fi}(T)|^2 = \frac{1}{\hbar^2} |\langle \Psi_f | \tilde{H}_{\text{int}} | \Psi_i \rangle|^2 \left| \int_0^T e^{i(\Delta E/\hbar - \omega_i)t} dt \right|^2 ,$$

where $\tilde{H}_{\text{int}} = e\,\mathcal{E}_i\, \boldsymbol{\epsilon}_i \cdot \mathbf{r}/2$ is the time independent part of the perturbation and ΔE is the energy difference of the 1s and 1p orbitals.

(a) First sum $|C_{fi}(T)|^2$ over the three final states by explicitly evaluating the three matrix elements. Make your life easier by choosing your coordinate system so that the z-axis is parallel to $\boldsymbol{\epsilon}_i$ and the three p-states have definite (even or odd) symmetry with respect to reflection in z. Your result, of course, is independent of coordinate system and p-state basis. Expressions for the orbital wave functions can be found in the solutions of assignment 4.

You should now have an expression that depends on the photon parameters like this:

$$\sum_f |C_{fi}(T)|^2 = \mathcal{E}_i^2 \, F(\omega_i) \; .$$

As a first step in summing over the initial photon states, consider just the sum over those photon states with frequency ω_i within a small range $d\omega$ about a particular ω. The function $F(\omega_i)$ may be treated as a constant over this range and the remaining sum has a clear physical interpretation:

$$\sum_{i\,:\,\omega<\omega_i<\omega+d\omega} \frac{1}{8\pi}\mathcal{E}_i^2 = u(\omega)\, d\omega \; .$$

Each electromagnetic wave (photon) with electric field amplitude $\mathcal{E}_i$ contributes, in cgs units, an amount $\mathcal{E}_i^2/8\pi$ to the electromagnetic energy density. We express the result in terms of the spectral energy density $u(\omega)$ (energy per volume per interval of frequency). This identification transforms the sum over **all** photon states into an integral over ω:

$$\sum_{f,i} |C_{fi}(T)|^2 = 8\pi \int u(\omega) F(\omega) d\omega \; .$$

(b) Evaluate the integral over ω in the limit of large T. As in the Golden Rule derivation, the integrand is strongly peaked at $\omega = \omega_0 = \Delta E/\hbar$ and grows linearly with T. Express your answer for the transition rate

$$\Gamma = \frac{1}{T} \sum_{f,i} |C_{fi}(T)|^2$$

in terms of ω_0, $u(\omega_0)$, the cavity radius R, and fundamental constants.

2. One proposal for reading out the state of the electrons-on-helium quantum computer is to apply a DC electric field normal to the helium surface that enables the electrons to tunnel from the surface to detectors above the helium. The sensitivity of the tunneling rate to the initial state of the electron, $|0\rangle$ or $|1\rangle$, makes this a very discriminating measurement of the quantum state of the computer.

   As in previous assignments, an electron in this device is subject to an effectively infinite barrier for $z < 0$ and an attractive image charge potential $-\lambda e^2/z$ for $z > 0$. Now there is, in addition, the potential $-e\mathcal{E}z$ due to a uniform electric field $\mathcal{E}$. Recall also that the energy levels (for $\mathcal{E} = 0$) have the hydrogenic form, $E_n = -(\lambda/n)^2 e^2/2a_{\mathrm{B}}$, where the qubit states $|0\rangle$ and $|1\rangle$ correspond to $n = 1$ and $n = 2$ respectively.

(a) In the WKBJ formalism, the wave functions in this problem have two turning points. Show that in the limit of small $\mathcal{E}$ these are given by

$$\begin{aligned} z_1 &= (2n^2/\lambda) a_{\rm B} \\ z_2 &= (\lambda^2/2n^2) \frac{e/\mathcal{E}}{a_{\rm B}} \ . \end{aligned}$$

(b) For small $\mathcal{E}$ the ratio z_1/z_2 is small and can be replaced by zero in your calculations. With this in mind, show that the local momentum in the tunneling region is given by

$$|p(z)| = \sqrt{\frac{2\lambda m e^2}{z_1}} \sqrt{(1 - z_1/z)(1 - z/z_2)}$$

(c) Now consider the WKBJ wavefunction

$$\Psi_n(z) \sim \frac{A_n}{\sqrt{|p(z)|}} \exp \frac{1}{\hbar} \left(- \int_{z_1}^{z} |p(z)| dz \right)$$

in the limit $z_1 < z \ll z_2$ (far from the right turning point) where we can make the approximation

$$\int_{z_1}^{z} \sqrt{(1 - z_1/z)(1 - z/z_2)}\, dz \approx \int_{z_1}^{z} \sqrt{1 - z_1/z}\, dz \ .$$

Evaluating the integral, we find the leading behavior

$$\int_{z_1}^{z} \sqrt{1 - z_1/z}\, dz = z - \frac{z_1}{2} \left(1 + \log\left(4z/z_1\right)\right) + \cdots$$

up to terms that vanish for $z_1 \ll z$. This must match, in the tunneling region, the hydrogenic bound state wave functions from the previous assignments ($\tilde{a}_{\rm B} = a_{\rm B}/\lambda$):

$$\begin{aligned} \Psi_1(z) &= 2(z/\tilde{a}_{\rm B}) \exp\left(-z/\tilde{a}_{\rm B}\right)/\sqrt{\tilde{a}_{\rm B}} \\ \Psi_2(z) &= \frac{1}{\sqrt{2}} \left(z/\tilde{a}_{\rm B} - \frac{1}{2}(z/\tilde{a}_{\rm B})^2 \right) \exp\left(-z/2\tilde{a}_{\rm B}\right)/\sqrt{\tilde{a}_{\rm B}} \\ &\sim \frac{-1}{2\sqrt{2}} (z/\tilde{a}_{\rm B})^2 \exp\left(-z/2\tilde{a}_{\rm B}\right)/\sqrt{\tilde{a}_{\rm B}} \ . \end{aligned}$$

Since these bound state wave functions are normalized, you can determine the normalization constant A_n of the WKBJ wave function. Find the values of A_1 and A_2.

(d) Use a connection formula to find the **normalized** WKBJ wave function for $z > z_2$. A key part of the normalization is the Gamow factor e^{-G}, where

$$G = C \left(\frac{\lambda}{n} \right)^3 (\mathcal{E}_0 / \mathcal{E}) \ ,$$

and $\mathcal{E}_0 = e/a_{\mathrm{B}}^2$. Derive this formula and determine the numerical constant C.

(e) Calculate the tunneling lifetime $\tau = 1/j(z)$ by evaluating the probability current $j(z)$ for your carefully normalized WKBJ wave function in the region $z > z_2$. Compare the lifetimes of the $|0\rangle$ and $|1\rangle$ states ($n = 1$ and $n = 2$) when $\mathcal{E} = 10$ V/cm.